Deflated Continuation in `pde2path`

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July 10, 2018

Abstract

`pde2path` is a tool for numerical calculation of solutions of partial differential equations. Bifurcation diagrams are created by numerical continuation which clarify the behaviour of solutions depending on a continuation parameter. Shooting type continuation can lead to errors, e.g. not all branches are detected or misbehaviour of the branches. To avoid these problems, a recently proposed bifurcation analysis technique, called *Deflated Continuation*, can be utilized. The resulting deflated problem is constructed via the application of a deflation operator to the residual and solved by an iterative process applied to the transformed problem to find other possible solutions for each fixed value of the continuation parameter. After analyzing different deflation operators theoretically, an example implementation in `pde2path` for the Allen-Cahn equation is presented.

Contents

1 Introduction

First we give a short introduction into the continuation MATLAB package `pde2path` and how the common way of continuation works. Later we show how to improve this method via deflated continuation and apply it exemplary on the Allen-Cahn equation. Therefore, we introduce the theoretical background of this equation also in this chapter.

1.1 `pde2path`

`pde2path` is a package for continuation of solutions of partial differential equations in MATLAB and is based on the `OOPDE` toolbox [10]. It treats PDE systems of the form

$$\partial_t u = \nabla(c \otimes \nabla u) + b \otimes \nabla u - au + f \tag{1}$$

where $u = u(x) \in \mathbb{R}^N$, $x \in \Omega \subset \mathbb{R}^2$, Ω some bounded domain, parameter $\lambda \in \mathbb{R}$, $c \in \mathbb{R}^{N\times N\times 2\times 2}$, $b \in \mathbb{R}^{N\times N\times 2}$, $a \in \mathbb{R}^{N\times N}$ and $f \in \mathbb{R}^N$ can depend on x, u, ∇u and parameters (especially on λ). The focus is on stationary equations

$$-G(u;\lambda) \coloneqq \nabla(c \otimes \nabla u) + b \otimes \nabla u - au + f \stackrel{!}{=} 0. \tag{2}$$

Also generalized Neumann boundary conditions of the form

$$\mathbf{n} \cdot (c \otimes \nabla u) + qu = g \tag{3}$$

are supported, where $\mathbf{n}$ is the outer normal and $q \in \mathbb{R}^{N\times N}$ and $g \in \mathbb{R}^N$ can depend on x, u and parameters [13].
The general idea of `pde2path` is finding all solutions of equation (2) by numerical continuation. The parameter λ is called the continuation parameter and we want to find solutions depending on λ.
The continuation function `cont` in `pde2path` is based on a predictor-corrector-method to solve the equation (2) numerically with respect to different values of the continuation parameter. The predictor-step starts from an already known solution (u_0, λ_0) and ends up on a manifold with a certain property, e.g. constant stepsize in λ (more details in [5],[3]). The corrector-step follows, usually done by newton's method with damping to find a solution (u_1, λ_1)

on this manifold. After checking for bifurcation points via bisection between (u_0, λ_0) and (u_1, λ_1) and plotting the new continuation point (u_1, λ_1) in a diagram, the predictor step starts again. In this way a branch of a bifurcation diagram is produced. If a bifurcation point is found during the continuation, i.e. there arises a new solution to our considered problem, a new solution branch will be found. We use the function `swibra` to switch to this new branch. By calling `cont` again, this new solution branch can be followed. This is how a bifurcation diagram is produced in `pde2path` [2].
For the basic idea of continuation and bifurcation, the algorithms and a more detailed explanation of equation (2) we refer to [3] and the references therein. Additionally for more information about the updated version `pde2path 2.0` and the improvements we refer to [13].

1.2 Allen-Cahn equation

We consider the one-dimensional Allen-Cahn equation

$$\partial_t u = \partial_{xx} u + u(\lambda - u^2) \tag{4}$$

with continuation parameter $\lambda \in \mathbb{R}$ on a symmetric domain with $x \in \Omega = [-L, L]$ and homogeneous Neumann boundary conditions

$$u_x(-L) = 0 = u_x(L)\,.$$

This reaction-diffusion equation was first introduced by Allen and Cahn in [1] to describe the motion of anti-phase boundaries, a planar crystallographic defect in crystalline solids. This defect occurs in ordered alloys and the concentration of one of the two metallic components of the alloy is represented by u. The Neumann boundary conditions ensure that there is no mass loss across the boundary walls. Today the Allen-Cahn equation, often considered as time-dependent Ginzburg-Landau equation, is commonly used in moving interface problems in material science and fluid dynamics in a phase-field approach (see e.g. [7]), furthermore the equation is often considered because of its energy stability [11].
The trivial solution $u^{(0)} \equiv 0$ is a stationary solution of equation (4), and by linearization we can analytically compute the bifurcation points (see [2])

$$\lambda_j = \left(\frac{j\pi}{2L}\right)^2 \quad \text{with eigenfunctions} \quad \nu_j(x) = \cos\left(\frac{j\pi}{2L}x\right) \quad j = 0, 1, 2, \dots\,.$$

The bifurcation points of the Laplacian are negative, but as already mentioned in section 1.1, we focus in `pde2path` on solutions of $-G(u, \lambda) = 0$, such that we actually compute the bifurcation points of the negative Laplacian.

2 Deflated Continuation

With the help of deflated continuation we want to improve the current algorithms in `pde2path` for producing bifurcation diagrams. In the following we will analyze this type of continuation in more detail.

2.1 Motivation

Consider a problem G with solution $u \in \mathbb{R}^n$ parameterized by a parameter $\lambda \in \mathbb{R}$

$$G(u, \lambda) = 0\,. \tag{5}$$

The number of solutions of (5) may vary for different λ. Analyzing the variation of a solution u with λ is the subject of bifurcation theory [9].
If a point $u^{(1)}$ is identified as a solution to (5), usually with iterative algorithms like Newton's method, such that $G(u^{(1)}, \lambda_1) = 0$, the implicit function theorem will ensure, under suitable conditions on G, the existence of open neighbourhoods around $u^{(1)}$ and λ_1 and a unique differentiable function g such that $G(u, \lambda) = 0$ implicitly defines a function $u = g(\lambda)$ in this neighbourhoods. So we are able to define solution curves which consists of points (u, λ) with $G(u, \lambda) = 0$. With numerical continuation these solution curves are traced out [9].
As already explained in section 1.1, this numerical continuation is usually done via predictor-corrector-methods like Newton's method, which is shown in figure 1.

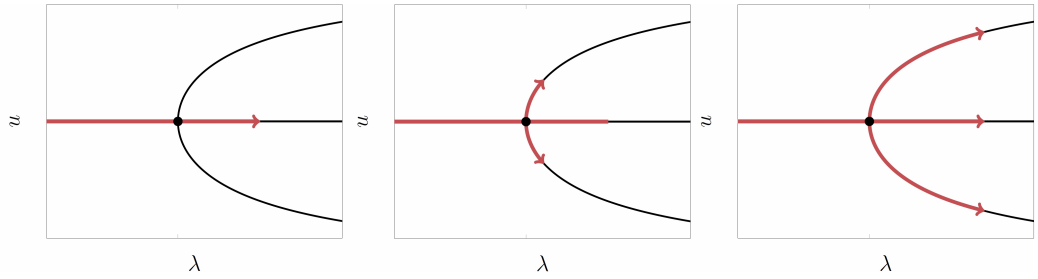

Figure 1: Numerical continuation by predictor-corrector-methods [8]: Continuation with an initial guess on a solution arc, loading known bifurcation points and again continuation.

But this type of numerical continuation does not work accurate for all problems. Only solutions which are continuously connected with the known initial solution will be found by using predictor-corrector-methods (see figure 3, left) [8]. Another issue would occur if the solution branches are very close to each

other. The corrector step could lead to unwanted branch switching by converging to a solution which is not on the considered branch. This would produce confusing and incorrect bifurcation diagrams (see figure 2).

The technique called *Deflated Continuation* solves these problems by adding a Deflation operator to the considered problem.

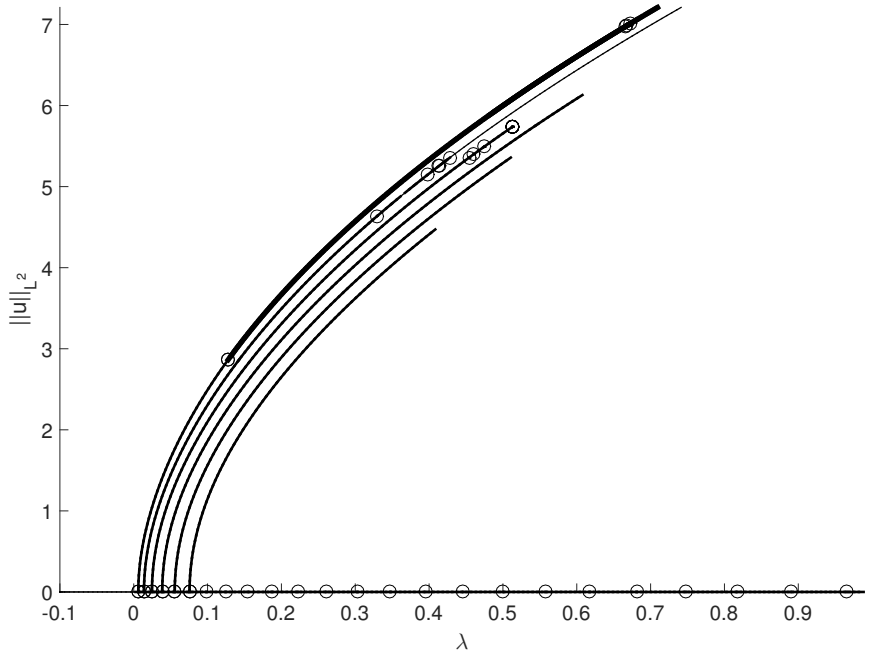

Figure 2: Bifurcation diagram of Allen-Cahn equation on $\Omega = [-40, 40]$ with homogeneous Neumann boundary conditions by numerical continuation in `pde2path`. The bifurcation points and the branches are close to each other because of the large domain, this can lead to undesired branch switching. The outermost branch should consist only the spatially homogeneous solutions (more detailed explanation in section 3), but first the continuation finds another branch and later it switches to the homogeneous one (thicker line).

2.2 Deflation operator

The number of solutions of (5) for a fixed parameter value λ is unknown and can be arbitrarily large. The current numerical continuation would only find one solution in one continuation step. To find another solution for the same λ, a suitable bifurcation point and many continuation steps are necessary,

because applying Newton's method again to the considered problem will only find the same solution. The possibility to find *all* solutions for a fixed λ in one continuation step would lead to the ability to build the bifurcation diagram with growing λ, i.e. "from left to right". In this way both disconnected branches will be found (see figure 3) and also branches close to each other will not blended.
The general idea of deflated continuation is now to transform the equation (5) such that only the already known solution $u^{(1)}$ is no longer a solution of the transformed equation. Now an iterative algorithm like Newton's method applied to this transformed equation, starting from an initial guess unequal $u^{(1)}$, should not converge to the known solution $u^{(1)}$ but to another solution of (5) if it exists. Using this, we are able to find all solutions of the problem for a fixed $\lambda \in \mathbb{R}$. This transformation is realized by adding a deflation operator to the residual (5) [9].

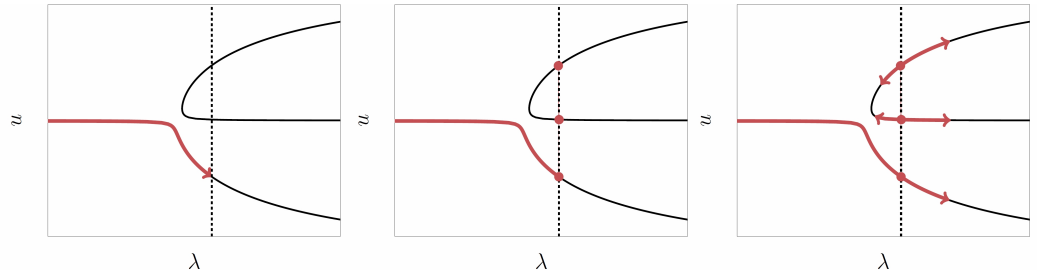

Figure 3: left: Problem with unconnected branches, middle/right: Solving problem by finding all solutions for a fixed λ [8].

Definition (*Deflation Operator* [9])
Let V, X, Z be Banach spaces, $U \subset V$ open. Be $G : U \longrightarrow X$ a differentiable operator and for all $u^{(1)} \in U$, $u \in U \setminus \{u^{(1)}\}$ be $W(u; u^{(1)}) : X \to Z$ an invertible linear operator. Then W is called **Deflation Operator** of G, if for $G(u^{(1)}) = 0$ and $G'(u^{(1)})$ non singular it holds:

$$\liminf_{i \to \infty} \|W(u_i; u^{(1)}) G(u_i)\|_Z > 0\,,$$

for every sequence $\{u_i\} \subset U \setminus \{u^{(1)}\}$ converging to $u^{(1)}$.
We define the shifted and exponentiated-normed deflation operator

$$W(u; u^{(1)}, p, \alpha) := \left(\frac{1}{\|u - u^{(1)}\|_p^p} + \alpha \right) \mathcal{I} \tag{6}$$

where $\mathcal{I}$ is the identity operator, $u^{(1)}$ the already known solution to (5) with $G'(u^{(1)})$ non singular, $\alpha \in \mathbb{R} \geq 0$ and $p \in \mathbb{N} \geq 1$ [9].

For a sufficient condition of identifying deflation operators and consequent the proof that (6) is a deflation operator, see [9].
Now applying Newton's method to the operator $W(u; u^{(1)}, p, \alpha)G(u, \lambda)$ will not converge to the already known solution $u^{(1)}$, but to all other roots of G (depending on the initial guess $u^{(0)}$). In the following we want to analyze the parameters α and p and the correlation to the initial guess.

2.3 The exponent p

Like in (6) defined, the exponent p needs to be greater than or equal one, because otherwise the already known root is still a root (with respect to the multiplicity of roots), and the considered p-norm would not exist.
To analyze the influence of the exponent p, we can assume first $\alpha = 0$ and consider the following simplified scalar example $f(x) = x(1 - x^2)$ with the known root $x^{(1)} = 0$. This results in the deflated function

$$F(x) = \frac{1}{|x|^p} x(1 - x^2)$$

which is plotted in figure 4 for different values of p. Observations of characteristics in this scalar deflation can be transferred to the general case.

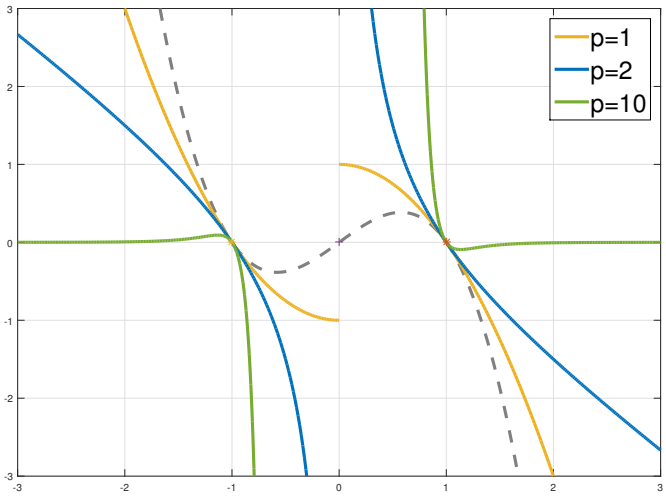

Figure 4: Deflated function F for $p = 1, 2, 10$ and $\alpha = 0$. Gray dashed line represents initial function $f(x) = x(1 - x^2)$.

Choosing a higher exponent p leads to faster growing of the term

$$\frac{1}{\|u - u^{(1)}\|_p^p}$$

near $u^{(1)}$. By choosing an initial guess near $u^{(1)}$, there will be slow convergence of Newton's method, because of the almost vertical tangent. So by choosing p high, the initial guess has to be away from the deflated root $u^{(1)}$ to achieve efficiently convergence behaviour.
Furthermore, two important values for p can be observed. If the value of p is less than the multiplicity of the deflated root, there will be only a removable singularity in the deflated root, so the deflation will not be successful. Also important is the smallest value of p from which it holds

$$\lim_{\|u\| \to \infty} W(u; u^{(1)}, p, \alpha) G(u, \lambda) = 0\,,$$

because this can lead to small residuals and false convergence of Newton's method. If the norm of the initial guess is too high, the residual could be already small enough for incorrectly reporting numerical convergence.
Also it can be observed in figure 4 that for higher p the scalar deflated function is not monotone anymore; new extrema occur. Newton's method starting in such points would fail, because of the singular derivative.
To find an optimal p for the considered problem in (5), until now we don't have a guidance for the choice, we recommend on numerical experimentation. For special functions there may be an optimal p computed, but this is not for the general case.

2.4 The shift parameter α

Like in (6) defined, the shift parameter α needs to be greater or equal zero, because otherwise we would produce an unwanted additional root at $\bar{u}$ with $\frac{1}{\|\bar{u}-u^{(1)}\|} + \alpha = 0$ by applying the deflation operator to any problem.
In the simplified scalar example above the influence and the need for this shift α can be clearly seen. Consider again $f(x) = x(1 - x^2)$ with the known root $x^{(1)} = 0$ and the resulting deflated function with $p = 10$

$$F(x) = \left(\frac{1}{|x|^{10}} + \alpha\right) x(1 - x^2)\,,$$

shown in figure 5 for different values of α.
Applying Newton's method to the deflated problem with $\alpha = 0$, the algorithm finds that the function value of the deflated function can be arbitrarily near

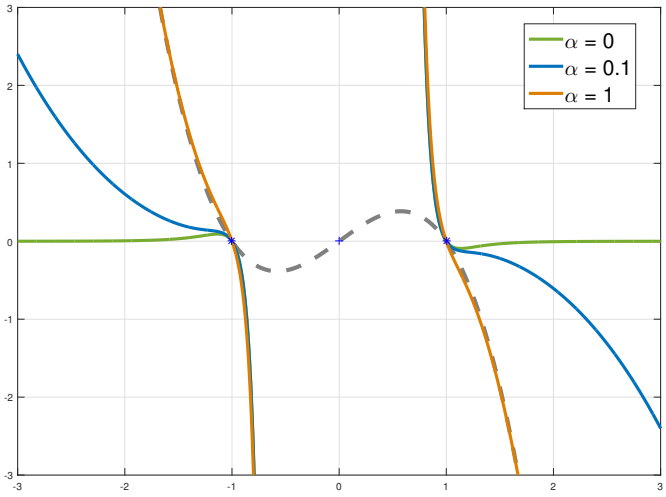

Figure 5: Deflated function F for $p = 10$ and $\alpha = 0, 0.1, 1$. Gray dashed line represents initial function $f(x) = x(1 - x^2)$.

to zero if x is going to be very small or very large, i.e. $x \to \pm\infty$, so the algorithm will report erroneously successful convergence. For $\alpha \neq 0$ there is no convergence to zero for $|x| \to \infty$, the problem of false convergence is solved.
The additional term $\alpha\mathcal{I}$ in the deflation operator in (6) shifts the deflated function away from zero for $\|u-u^{(1)}\|_p \to \infty$, because if $\|u-u^{(1)}\|_p^p$ is growing faster than $\|G(u,\lambda)\|$ for u far away from $u^{(1)}$ it holds

$$W(u, u^{(1)}, p, \alpha)G(u,\lambda) = \frac{G(u,\lambda)}{\|u - u^{(1)}\|_p^p} + \alpha G(u,\lambda) \approx \alpha G(u,\lambda)$$

The problem mentioned in section 2.3 about false convergence of Newton's method for p too high is solved by this shift parameter α [9].
For $\alpha = 1$ the deflation operator is nearly the identity away from the deflated root, which can also presented in figure 5. This leads to suggest an default value $\alpha = 1$, but in [9] it is discussed that numerical experience at an explicit example yields to benefits for choosing other values for α. The conclusion there is that we are unable to give a guidance for the choice of the shift parameter such that this is the best choice for general problems, again numerical experimentation is recommended [9].
It is also observable in figure 5 that through the choice of $\alpha \neq 0$ the extreme points disappear. This fact can be utilized to avoid new extrema by change

of the parameter p. Therefore, suitable combinations of α and p are needed for individual problems.

2.5 More solutions

The deflation operator in (6) can also be utilized to find further solutions in addition to a known one. But the goal is to find *all* solutions for a fixed λ to construct the whole bifurcation diagram.
If the solutions $u^{(1)}, \ldots, u^{(n-1)}$ are already found, an additional n-th solution (if it exists) can also be found by the use of (6)

$$W(u; u^{(1)}, \ldots, u^{(n-1)}, p, \alpha) \coloneqq \left(\frac{1}{\prod_{i=1}^{n-1} \|u - u^{(i)}\|_p^p} + \alpha \right) \mathcal{I}\,.$$

This idea was already used in 1963 by J. H. Wilkonson to find roots of polynomials (see [4]) and picked up by P. E. Farrell in 2015 (see [9]).
With this kind of deflation operator we do not have to choose the value of p necessarily greater than the multiplicity of the deflated solution like mentioned in section 2.3 (because in general we do not know this multiplicity). We can find the same solution again and deflate it until a real singularity results. As an example, consider $f(x) = x^3$ and the known solution $x^{(1)} = 0$. By choosing $p = 1$ lower than the multiplicity of the known solution, $x^{(1)} = 0$ is deflated with a removable singularity. Now Newton's method can find $x^{(2)} = 0$, and and after deflating this root too, $x^{(3)} = 0$ can also be found. No other roots of f exist, all roots were found by the deflation technique, even if p was chosen smaller than the multiplicity.

2.6 Optimality

One main question is if there exists an optimal deflation operator (i.e. optimal p and α) and an initial guess, depending on the choice of the parameters p and α, such that Newton's method will converge fast, with respect to a minimal number of iteration steps?
Like in sections 2.3 and 2.4 already mentioned, there exists no universal choice for the parameters p and α which is optimal, numerical experimentation is recommended. Also there is no problem independent possibility to choose an optimal initial guess $u^{(0)}$, but for a concrete problem $G(u, \lambda)$ the initial guess could be computed if α and p are given.
As an example, take the one dimensional Allen-Cahn equation in (4) and focus on stationary and spatially constant solutions such that the related

problem is given by

$$G(u, \lambda) = u(\lambda - u^2) \stackrel{!}{=} 0\,. \tag{7}$$

In theory, the occurring bifurcation is called a *supercritical pitchfork bifurcation*, where a transition from one to three real equilibria happens by changing the parameter λ. For negative λ there is only one (linearly) stable equilibrium at $u \equiv 0$, for positive λ there exists an unstable equilibrium at $u \equiv 0$ and two stable equilibria at $u \equiv \pm\sqrt{\lambda}$. For more details about bifurcation theory see e.g. [6].

For $\lambda \leq 0$ with $u^{(1)} \equiv 0$ we already know all solutions for our problem (7), so we analyze the case $\lambda > 0$. Now we want to find new solutions by applying Newton's method with an initial guess $u^{(0)}$ to the deflated problem $F(u; u^{(1)}, p, \alpha, \lambda) \coloneqq W(u; u^{(1)}, p, \alpha)G(u, \lambda)$.

Newton's method (without damping) is defined as

$$u^{n+1} = u^n - \frac{F(u^n; u^{(1)}, p, \alpha, \lambda)}{F'(u^n; u^{(1)}, p, \alpha, \lambda)} \quad n \in \mathbb{N}_0\,. \tag{8}$$

The goal is now to find an optimal initial guess u^0 such that Newton's method needs a minimal number of iteration step until convergence.

In the following we want to analyze and compare the cases $p = 2, 3, 4$ with $u^{(1)} = 0$ by finding an optimal initial guess u^0 in (8). Choosing u^0 too small leads to very high gradients in the first iterations of Newton's method because of the singularity at zero and following from this to a small stepsize, this leads to slow convergence. Choosing u^0 too big we might be too far away from the other roots and Newton's method will not converge within a reasonable number of iterations.

Because of symmetry, we can assume to choose u^0 positive to find the positive solution $\sqrt{\lambda}$, all results can be transferred to negative u^0 to find the negative solution $-\sqrt{\lambda}$.

The considered function $F \in C^\infty(0, \sqrt{\lambda}]$ is monotonically decreasing and convex on the interval $(0, \sqrt{\lambda}]$, so newton's method converges for all $u^0 \in (0, \sqrt{\lambda}]$ monotonically from the left [12]. This means, within a Newton step, we can not reach an $u^0 > \sqrt{\lambda}$, hence we can not take a step "too big". To find the optimal initial guess we want to find such u^0 which leads to the largest value of $\|u^1 - u^0\|$ after the first iteration step.

Because we are only interested in the interval $(0, \sqrt{\lambda}]$, possible convergence to zero for $\|u\| \to \infty$ must not be heeded, we can choose $\alpha = 0$ (even if this is not the assured optimal choice). From (8) we get that u^0 leads to biggest stepsize in the first iteration step if

$$u^0 = \operatorname*{argmax}_u \left(-\frac{F(u; u^{(1)}, p, \alpha, \lambda)}{F'(u; u^{(1)}, p, \alpha, \lambda)} \right)$$

In the presented exemplary case, the results are given in table 1.

p	u^0	stepsize	resulting u^1
2	$0.486\sqrt{\lambda}$	$0.3\sqrt{\lambda}$	$0.786\sqrt{\lambda}$
3	$0.577\sqrt{\lambda}$	$0.193\sqrt{\lambda}$	$0.770\sqrt{\lambda}$
4	$0.628\sqrt{\lambda}$	$0.146\sqrt{\lambda}$	$0.774\sqrt{\lambda}$

Table 1: Results for optimal initial guess u^0 for different values of p and for $\alpha = 0$.

For higher values of p, the initial guess has to be closer to the root $\sqrt{\lambda}$ we seek for, but the resulting next iterated is not necessary closer. It can be recognized that choosing higher p worsens the stepsize of the first iteration step. This leads to the conclusion that $p = 2$ could be the optimal choice in this case. Furthermore, it should also be noted here (cf. 2.3) that increasing the parameter p leads to a "blow-up" of the singularity $u^{(1)}$, which at the same time corresponds to an increase in the choice of u^0.

3 Implementation in `pde2path`

The implementation of the one-dimensional Allen-Cahn equation (4) on the domain $\Omega = [-4, 4]$ and with homogeneous Neumann boundary conditions in `pde2path` leads to the following bifurcation diagram which is illustrated in figure 6 by "normal" numerical continuation (for a documentation of the original implementation of this problem see [2]). The branches starting from the bifurcation points include solutions similar to their associated eigenfunctions, e.g. the first branch includes only spatially constant solutions. Because of the symmetry in the considered problem, one solution in bifurcation diagram belongs to two solutions in the equation.
Now the implementation of deflation in `pde2path` with the goal to built the bifurcation diagram by deflated continuation is discussed. For the moment this implementation is very straightforward and not finished yet.
The information about the considered problem are implemented in the files `sG` and `sGjac` (the operator G and its jacobian). By saving the already known solutions and their number which should be deflated as `p.deflate.uold` and `p.sw.deflate` respectively, the deflation can be implemented in `sG.m` and `sGjac.m` as a weight function multiplied with the original residual. This weight is calculated depending on `p.sw.deflate`.
After numerical continuation of the trivial zero solution in `pde2path` as usual (`p.sw.deflate=0`), testing the first deflation was done by the following routine. A point on the trivial solution arc with a value of $\lambda > 0$ was loaded

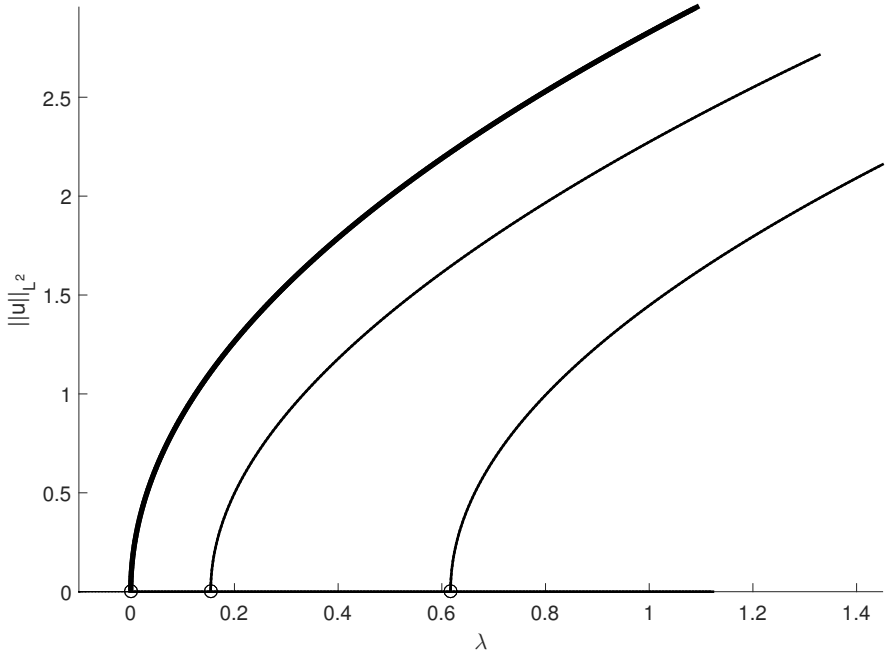

Figure 6: Bifurcation diagram for the one-dimensional Allen-Cahn equation on $\Omega = [-4, 4]$ with homogeneous Neumann boundary conditions generated via `pde2path`.

and saved as solution to be deflated in `p.deflate.uold`. Then the trivial solution got perturbed by an eigenfunction ν_j, deflation was switched on by `p.sw.deflate=1` (deflation of one known solution), the perturbed solution was used as initial guess and finally Newton's method results (hopefully!) in another solution – on the branch corresponding to the used eigenfunction – for the same value of λ. To find more solutions (see section 2.5), the procedure is analogous, the found solution has to be saved also and the initial guess is the trivial solution perturbed with a multiple of the eigenfunction.

4 Numerical results (Allen-Cahn)

Depending on the value of λ in the loaded point, there exist a different number of solutions (cf. section 1.2). Of course with deflation we want to find all possible solutions, and preferably with a minimal number of iterations

of the Newton's method.

Exemplary we load a solution on the trivial solution arc with $\lambda = 0.9949$ (see figure 6 for an illustration). By computing the bifurcation points from section 1.2 we get three previous bifurcation points for λ equal to 0, 0.154, and 0.617, so six solutions (except the trivial one) are expected (we know this from the course of the solution branches in the bifurcation diagram in figure 6). Table 2 shows the number of solutions found via deflation, the multiplication factor of the experienced optimal perturbation with the eigenfunctions ν_j (ν_j belongs to the j-th solution arc from the outside to the inside) and the numbers of iterations for every found solution for different choices of p and α. To compare the numbers of iterations correctly, Newton's method without damping is used.

# solutions	p	α	perturbation $\cdot[\nu_3, \nu_2, \nu_1]$	# iterations
3	1	0	[1.4 -1.4 2.4]	[3 2 2]
6	1	0.1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]
6	1	1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]
3	2	0	[1.4 -1.4 2.4]	[3 2 2]
6	2	0.1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]
6	2	1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]
1	10	0	[1.4]	[3]
6	10	0.1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]
6	10	1	[1.4 -1.4 2.4 -2.4 2.8 -2.8]	[3 3 4 4 3 3]

Table 2: Numerical results of deflation for $\lambda = 0.9949$

It is obvious that the choice of $\alpha = 0$ is not recommendable because in this cases never all solutions were found. Also the interval of multiplication factors for perturbation which leads to convergence is not big in this cases (only ± 0.4 around the optimal value), in contrast to the cases $\alpha \neq 0$, where the interval of multiplication factors which leads to convergence is unlimited, only a choice too close to zero will lead to convergence to the trivial solution again. Apart from that, we did not recognize any differences between the cases for different p but $\alpha \neq 0$.

5 Outlook

After theoretical analysis of the characteristics of deflation operators and defining our specific operator, the implementation of Deflation for the Allen-Cahn equation followed. For fixed λ we are able to find all solutions with the

help of this Deflation operator.
Besides some questions have remained open. First, we would like to know if other iteration schemes may lead to better convergence rates? Also we implemented the problem not only for $\|\cdot\|_p^p$ as norm in the deflation operator but also for $(\|\cdot\|_2^2)^p$ and got similar results. Will other norms lead to different results? Moreover, in sections 2.3 and 2.4 in the scalar example is mentioned that for different p and α extreme points arise, where Newton's method would fail. Is there a correlation between the choice of p and α and this extreme points?
As already mentioned, the implementation is also not finished yet. The goal is to construct a whole bifurcation diagram by deflated continuation. As described in section 3 we only implemented the deflation in one point. The next step would be the automation of this process for constructing the whole bifurcation diagram. Additionally, we would prefer the implementation in the predictor-step of Newton's method as an universal solution. Also the optimal perturbation of the discussed problem was only approximately computed with numerical experimentation. This could also be automated. And finally, until now we only implemented the deflation operator to the Allen-Cahn equation, the implementation for other problems is still open.

References

[1] S.M. Allen and J.W. Cahn. "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening". In: *Acta Metall. Mater.* 27 (1979), pp. 1085–1095.

[2] J. Rademacher H. Uecker. "The OOPDE setting of pde2path - a tutorial via some Allen-Cahn models". In: (2017).

[3] J. Rademacher H. Uecker D. Wetzel. "a Matlab package for continuation and bifurcation in 2D elliptic systems". In: *Numerical Mathematics: Theory, Methods, Applications* (2014).

[4] J.H.Wilkonson. "Rounding errors in algebraic processes". In: *vol 32 of Notes on Applied Science* (1963).

[5] I.C.F.Ipsen K.I.Dickson C.T.Kelley and I.G.Kevrekidis. "Condition estimates for Pseudo-arclength continuation". In: (2006).

[6] H. Kielhöfer. *Bifurcation Theory*. Springer Verlag, 2012.

[7] J.Shen L.Q.Chen. "Applications of semi-implicit Fourier-spectral method to phase field equations". In: *Computer Physics Communications* 108 (1998), pp. 147–158.

[8] C.H.L.Beentjes P.E.Farrell Á.Birkisson. "The computation of disconnected bifurcation diagrams". In: (2016).

[9] S.W.Funke P.E.Farrell Á.Birkisson. "Deflation techniques for finding distinct solutions of nonlinear partial differential equations". In: (2015).

[10] Uwe Prüfert. *An object oriented toolbox for finite elements in Matlab.* URL: `http://page.math.tu-berlin.de/~pruefert/`.

[11] J. Shen and X.Yang. "Numerical approximations of Allen-Cahn and Cahn-Hilliard equations". In: *Discrete and cont. dynam. systems* 28.4 (2010), pp. 1669–1691.

[12] H. Späth. *Numerik: Eine Einführung für Mathematiker und Informatiker.* Springer Verlag, 2013.

[13] D. Wetzel T. Dohnal H. Uecker and J. Rademacher. "faster FEM, multi-parameter continuation, nonlinear boundary conditions, and periodic domains - a short manual". In: (2014).